Proceedings of the 4th International Conference on Educational Innovation and Philosophical Inquiries
DOI: 10.54254/2753-7064/9/20231117

For Mathematical Truth to Be Known: A Provability-based Condition Construction

Zhiru Lin[1,a,*]

[1]Shanghai World Foreign Language Academy, Shanghai, China
a. zhirulin_2024@stu.swfla.org
**corresponding author*

Abstract: One of the defining characteristics of mathematics is its emphasis on proof, which is the process of demonstrating the truth or falsehood of a mathematical claim through a rigorous and logical argument. However, while the concept of proof is central to mathematics, it is also a complex and multifaceted notion that raises a range of philosophical and practical questions. This article proposed a condition beyond the JTB framework for mathematical knowledge by using mathematics provability with a careful reflection on Gettier cases.

Keywords: justified true belief theory, provability, mathematical knowledge

1. Introduction

Mathematics is a fundamental branch of knowledge that has been studied and practiced for centuries. From ancient times to the present day, mathematicians have sought to understand the underlying structure and patterns of the natural world through abstract reasoning and logical deduction. One of the defining characteristics of mathematics is its emphasis on proof, which is the process of demonstrating the truth or falsehood of a mathematical claim through a rigorous and logical argument. However, while the concept of proof is central to mathematics, it is also a complex and multifaceted notion that raises a range of philosophical and practical questions. This is an essay about mathematical knowledge. This article would consider mathematical knowledge in light of Benacerraf's dilemma and Gettier cases. While it does not claim to have all the answers, the notion of *provability* as a potential solution to these problems will be provided.

2. Descriptive Knowledge

"Descriptive knowledge" is a term that is used to distinguish it from other types of knowledge, such as "procedural knowledge" or "know-how" (i.e., knowledge of how to do something) and "prescriptive knowledge" (i.e., knowledge of what one ought to do). In this context, "descriptive knowledge" refers to knowledge that is based on the description or representation of a fact, concept, or idea. The *justified true belief (JTB theory)* is an example of descriptive knowledge because it focuses on the properties that a belief must have in order to be considered knowledge [1]. Specifically, the JTB theory holds that a belief is knowledge if it satisfies the conditions of being justified, true, and believed by the person holding the belief [2-4].

DOI: 10.54254/2753-7064/9/20231117

"Descriptive knowledge" is used to describe a specific type of knowledge that is based on the description or representation of a fact, concept, or idea. This article will consider the notion of descriptive knowledge and the problem of Edmund Gettier cases [5]. Descriptive knowledge is commonly defined as *justified true belief (JTB theory)*[1], which holds that:

S knows that p if and only if

(i) S believes that p,

(ii) p is true, and

(iii) S is justified in believing that p.

This definition aims to exclude false beliefs, superstition, lucky guesses, and erroneous reasoning. "Erroneous reasoning" is excluded from the JTB theory because it can lead to false beliefs. The JTB theory aims to provide a reliable method for determining what constitutes knowledge, and it does so by requiring that a belief must be justified and true in order to be considered knowledge. However, if the justification for a belief is based on faulty or erroneous reasoning, then the belief may still be false even if it seems justified to the person holding the belief. Therefore, in order to avoid false beliefs, erroneous reasoning is ruled out as a valid form of justification for knowledge.

The concept of justification, as understood by the JTB theory, is internalistic, meaning that it is another mental state of the person, such as a perceptual experience, a memory, or a second belief in addition to the plain true belief. The justification supports the proposition and provides evidence or a reason for it. It is not the task for this study to compare the internalism vs externalism debate, however, to bear in mind that JTB theory proposes an internalistic concept is necessary.

Edmund Gettier proposed a series of cases in which a justified true belief does not amount to knowledge because its justification is not relevant to its truth, a problem commonly referred to as "epistemic luck" since the justification is merely a fortuitous coincidence. To illustrate this problem, consider the following example: it is noon right now, the clock points out that it is noon, and Josh looks at the clock and believes that it is noon. However, the clock is actually broken and always points to twelve o'clock. According to the JTB theory, Josh has acquired the knowledge that it is noon at present, but the justification for his belief is by luck.

To address the Gettier problem, various additions have been proposed, with the most widely accepted being the "causal theory" by Alvin Goldman [6]. This theory posits that a causal connection is essential to knowledge, and that the reasons for believing that a proposition is true must appropriately track whatever it is that makes the proposition true. Simply having good reasons to believe that a proposition is true is not sufficient for knowledge, as knowledge cannot be a matter of good luck. In the above counterexample, the causal theory asserts that it is the fact that it is noon that causes the clock to point to twelve o'clock and thus leads to Josh's belief, which should eliminate the luck issue in such cases.

3. Challenge in the Acquirement of Mathematical Knowledge

However, the causal theory is not without flaws. As Goldman himself noted, it applies only to empirical knowledge where the subject matter is concrete and causal. Philosopher Paul Benacerraf's dilemma further complicates matters by raising the question of how humans can acquire mathematical knowledge [7]. As concrete, spatiotemporally located beings, humans tend to obtain knowledge perceptually or through inference based on sensory experience. However, in the realm of mathematics, objects such as sets and numbers are abstract, non-spatiotemporal, and mind independent. As a result, traditional causal chains cannot be formed in the same way as they can for concrete or empirical knowledge. This raises the challenging question of how mathematical facts can be causally connected to the grounds on which we have inferred their truth. In short, can humans truly acquire mathematical knowledge?

DOI: 10.54254/2753-7064/9/20231117

4. Proof

This study suggests that mathematical truth is knowable to human beings, but an additional condition should be involved to form a link between the spatiotemporal human world and abstract mathematical objects. This additional condition is proof.

> *A mathematical proof is an inferential argument for a mathematical statement, showing that the stated assumptions logically guarantee the conclusion. The argument may use other previously established statements, such as theorems; but every proof can, in principle, be constructed using only certain basic or original assumptions known as axioms, along with the accepted rules of inference.*

This means proofs should be highly reliable. Any mistake will lead to a perjury. Furthermore,

> *Proofs are examples of exhaustive deductive reasoning which establish logical certainty, to be distinguished from empirical arguments or non-exhaustive inductive reasoning which establish "reasonable expectation".*

Each proof is required to be carried out step by step, with deductive logic connecting the steps. It is due to the existence (or the potential to exist concretely) of proof that humans judge the truthfulness of propositions.

In mathematical reasoning, axioms are taken as the starting point for the derivation of theorems. Axioms are basic assumptions that are considered self-evident and not requiring proof.

For example, Euclid's axioms are the basis of his geometry, and include statements such as "two points determine a straight line" and "all right angles are equal". These axioms are accepted as true without requiring any further justification and are used to derive other theorems within Euclidean geometry.

In terms of how axioms can be known, they can be arrived at through various means such as intuition or observation. For example, the axioms of Euclidean geometry are based on our perception of the physical world, where we observe straight lines, right angles, and parallel lines. Similarly, the axioms of set theory are based on our intuition of sets, where we consider sets to be collections of objects.

Therefore, while axioms are not derived through proof, they can be known like any other empirical knowledge through perception or intuition. If a mathematical statement (excluding the axioms, which can be known as any other empirical knowledge by perception or so) is not proved rigorously, then it cannot be confirmed as true or false. For example, Goldbach's conjecture, though it sounds plausible and has been tested with large numbers without any counterexamples, cannot be defined as true instead of a conjecture since no rigorous proof has been found yet. This is to say that a mathematical proposition (axioms excluded) is considered true to human beings if and only if it has the potential for a proof.

However, there might be a truth that isn't even in principle provable. This is known as an unprovable truth. In fact, there are known to be mathematical statements that are true, but for which no proof can exist. For example, Gödel's incompleteness theorems state that any formal system of mathematics that is powerful enough to represent arithmetic must contain statements that are true but unprovable within the system itself. Such statements are said to be independent of the system.

Therefore, it is important to note that the statement "a mathematical proposition (axioms excluded) is considered true to human beings if and only if it has the potential for a proof" applies only to those mathematical propositions for which a proof can exist. For unprovable mathematical statements, it may be more accurate to say that they are considered to be true based on their consistency with other accepted mathematical statements and principles, but they cannot be proven true through deductive reasoning.

When we say that proofs exist spatiotemporally and are concrete, we mean that they exist in a specific location and time, and they can be physically instantiated or written down in some form. For example, a mathematical proof can be written in a book or a computer file, or it can be

DOI: 10.54254/2753-7064/9/20231117

presented on a blackboard during a lecture. This makes proofs causal, in the sense that they can be traced back to their source and their construction can be explained in terms of a causal chain.

Moreover, proofs provide a link between the abstract world of mathematics and the concrete world of human experience. A proof allows us to demonstrate that a mathematical statement follows logically from a set of axioms and previously established theorems. This logical connection is what justifies our belief in the truth of the statement. Therefore, in the human world, proof can serve as a reference for mathematical truth, excluding the axioms which are taken to be true without requiring a proof.

In summary, the causal nature of proofs allows us to trace their connection to the axioms and previously established theorems, and the existence of a proof provides a reliable reference for mathematical truth.

5. How Proof Plays a Role in the Acquirement of Mathematical Knowledge

A proof, or the provability, should cause the individual's justification to the belief in mathematics, of an individual knowledge acquirer, referring to the truth itself.

There seems to be several different cases of how the proof can lead to the justification and then the belief. Let us consider some illustrative examples:

Situation 1:

(1) Mathematical claim *p* is true.

(2) Mathematician Smith proved by a rigorous process that *p* is true.

(3) Smith wrote the procedure of the proof and the result, which is the truth *p* down and publish it.

(4) A student Jones read the paper of proof and believes in the fact *p* as well as how to prove it.

Here, Smith discovers the proof of mathematical claim p, and further verified the truthfulness of *p*. The proof, in the form of a paper written by Smith, spread. Jones saw the paper which gave rise to his belief in *p* as a justification. This is a causal chain which relates Jones's belief in *p* back to the paper of proof for *p*, then to Smith's being able to prove *p* , and eventually to the *provability* of *p* itself, where the *provability* of *p* stands for the truthfulness of mathematical claim *p*. Therefore, Jones is said to be knowing *p* as his belief justified is causally related to mathematical truth *p*.

Exactly, in this situation, the proof serves as a bridge between the abstract mathematical truth and the concrete, spatiotemporal world. The paper of proof written by Smith is a concrete object that can be read and understood by Jones, who can then grasp the truth of p and its justification through the deductive steps in the proof. The causal chain is clear, as the proof links the truth of p to Smith's ability to prove it, and then to Jones's belief in it.

Situation 2:

Mathematical claim *p* is true

(1) Mathematician Smith proved by a rigorous process that *p* is true.Smith wrote the procedure of the proof and the result, which is the truth *p* down and publish it.

(2) Jones's teacher read the paper of proof and believes in the fact *p* as well as how to prove it.

(3) The teacher told Jones the fact *p* but he did not tell Jones how to prove it.

(4) Jones believes in fact *p*.

In this case, Smith discover the *provability* of mathematical claim *p*, and further verified the truthfulness of *p*. The *provability*, in the form of a paper of proof written by Smith spread and cause Jones's teacher to believe in *p*. Then, the teacher told Jones the mathematical claim *p* , also due to fact *p*'s proof. This further leads to the believing in *p* by Jones, which has a justification causally connected from the teacher, the paper, the proof, the *provability of p* which is a reference of claim *p* itself. The causal link between Jones and his teacher by *p's provability* should be ensured. This can

DOI: 10.54254/2753-7064/9/20231117

be achieved to assume that if Jones once ask, the teacher shall be able to give the proof to justify *p's provability*. Jones knows that *p*.

Kripke's causal theory of names could be relevant here [8]. In this theory, the reference of a name is determined by a causal chain linking the initial baptism of the name to the object it refers to. This is similar to how the justification of a belief in mathematics can be linked to the probability of a mathematical claim through a causal chain.

By following the causal chain, we can trace the justification of Jones's belief in p back to the provability of p, just as we can trace the reference of a name back to its initial baptism. However, it's important to note that the causal theory of names is specific to reference, while the causal link between the probability of a mathematical claim and the justification of a belief in mathematics is specific to knowledge.

Situation 3:

(1) Mathematical claim *p* is true.

(2) Mathematician Smith proved by a rigorous process that *p* is true.

(3) Smith did not write *p* down, but he published this result *p* claim that it has a proof.

(4) Jones read the paper and formed a belief of *p*.

In this case, Smith do not give out the proof concretely. However, it is not to say that proof do not exist. Though the proof do not seem to be causal since its non-spaciotemporal, it is still not to over-thrown our previous statement. Here, the causal connection similar to the one in the previous case, *provability* is what this article wants to introduce.

6. A New Terminology: *Provability*

Provabillity is the property that something, or more specifically a proposition, can be proved to be true. Though it is not to say that the proof should be already written down on a paper or so, there shall exists someone who has the ability of proving it at the very beginning of the knowledge spreading process. In other word, provability is the potential of a proposition to have at least one proof, and it is this proof, spatiotemporally or not, causes the justifications and triggers the belief.

Thus, the previous case is to be come up with the connection formed. Smith is able to proof mathematical claim *p*, meaning he discovers the *provability* of *p*. p's provability caused the publication of the paper about the claim itself. This further led to the belief by Jones, which is justified by the paper of claim. This is how the causal connection is formed. Jones's direct justification of belief is the paper by Smith, who should give out the proof to show *p*'s provability if being asked. If it is unable for Smith to give a concrete written proof, he shall be able to show the proof by any other method. For example, if the proof process is too long that a person cannot write them all down during his life span, there should be somehow a computer or so that can be programmed to give out the whole proof. This means it is still *p*, by its provability, cause the belief of Jones. *P* is thus known by Jones.

6.1. *Provability* and Truthfulness

Provability and truthfulness are both properties of a mathematical claim, but they differ in important ways. The truthfulness of a mathematical statement is objective, a priori, and independent of human calculation or reasoning. On the other hand, the probability of a statement depends on the work of human researchers who use rigorous methods to demonstrate its truth.

Despite these differences, provability is a necessary and sufficient condition for humans to judge the truthfulness of a mathematical claim. Without a proof, we cannot be certain whether a statement is true or false. However, if a proof is too obvious or boring, it may not have been written down or published. However, this does not mean that the statement is true or false without a proof. It simply

DOI: 10.54254/2753-7064/9/20231117

means that there is no currently available proof for the statement. In such cases, mathematicians may rely on intuition or common sense to make a judgment about the truthfulness of the statement, but without a proof, they cannot be certain. It is always possible that a counterexample or a flaw in the reasoning could be discovered later on, which would invalidate the initial judgment. Therefore, a proof is always necessary to establish the truth or falsity of a mathematical statement with certainty. Thus, provability serves as a reliable indicator of truth for human beings.

One might wonder whether a proof could be misleading or mistaken, leading us to believe a false claim is true. However, such events are unlikely in mathematics. Unlike concrete knowledge acquisition, such as the "clock" example mentioned earlier, proof and mathematical truth are not like a clock and time. For a mathematical claim to be provable, the proof must be free of error and highly rigorous, eliminating the possibility of luck or chance playing a role.

6.2. Axioms

An axiom is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments [9]. Mathematical axioms are a special type of mathematical claim that needs to be distinguished from ordinary ones. They serve as the foundation upon which other mathematical statements are logically derived. In other words, axioms are simply assumptions made by mathematicians on which they base their calculations and proofs. As a result, the truthfulness of axioms is not always certain, and they cannot be proven. There is no proof for an axiom, and thus our previous method of acquiring mathematical knowledge cannot be applied to these claims.

As for the axioms, they are considered to be the fundamental condition of all mathematical activities. Without them, no mathematical claim can be deduced or proved. This means that the mathematical statements that are thought to be known by most of us are actually uncomplete without the presence of these axioms as conditions. For instance, most people thought that "2+2=4"is a knowledge of theirs. However, it is in fact "if the axioms of arithmetic are true, then 2+2=4" To further dip in, *if-thenism* is to be consulted.

If-thenism is a philosophical perspective that views all claims, including mathematical claims, as conditional statements. It states that any claim can only be considered true if it is accompanied by its necessary conditions. In the case of mathematical claims, the necessary conditions are the axioms.

To illustrate this, let's consider the claim "2+2=4". According to if-thenism, this claim is actually "if the axioms of arithmetic are true, then 2+2=4". In other words, the truthfulness of the claim "2+2=4" is conditioned upon the truthfulness of the axioms of arithmetic.

So, axioms are not only necessary for the derivation of other mathematical claims, but they are also necessary for the establishment of the truthfulness of any mathematical claim. Without the axioms, mathematical claims are incomplete and cannot be considered true or false. They simply lack the necessary conditions for their truthfulness.

Therefore, while axioms cannot be proved or verified through mathematical reasoning, they are still essential components of the mathematical system. They serve as the foundation upon which all other mathematical claims are built and upon which their truthfulness is conditioned.

6.3. Generalization

As Benacerraf's dilemma put forward the doubt that whether *abstract* mathematics can be known by *concrete* human beings. It supposes that this problem does not exist only to the knowledge of mathematics, but also among all *abstract* knowledge. Here, as briefly mentioned before, *abstract* refers to outside space-time existence, non-causal, non-empirical and universal, examples include feelings, numbers and so on. While *concrete* objects, on the contrary, are located in space-time,

DOI: 10.54254/2753-7064/9/20231117

causal, empirical and particular. The problem seems similar to the previous one discussed about mathematics: Can these *abstract* things be able to be known by *concrete* human and if so, how?

Something similar to the mathematical *provability* shall be add to the acquirement process of such *abstract* knowledge. The terminology shall be, just like the discussed *provability*, rigorous where no counterexamples, luck effect or perjury are to be included, logical, and being strongly linked with truthfulness. Therefore, a very similar process to which by mathematical *provability* is to be applied. The causal chain can first link to the *provability-like* terminology and then establish to the truth itself.

However, finding a universal property that can apply to all abstract areas remains a challenge, and further effort is needed in this regard. Michael Dummett here, part of whose project involves finding concrete, empirical counterparts to notions like *proof*. Dummett's project, which seeks to find empirical counterparts to abstract notions such as *proof*, could be a useful approach in addressing this challenge [10].

7. Conclusion

In summary, this study proposed a condition beyond the JTB framework for mathematical knowledge, which is *provability*. This property provides mathematical truths with a potential proof that can justify our beliefs. In the absence of Gettier cases, the missing link in transforming mathematical truth into mathematical knowledge is a causal connection. However, due to the non-causal nature of mathematics, people must introduce external *provability* as a means of providing internal justification and enabling us to acquire mathematical knowledge. Therefore, the *provability* of mathematical claims is crucial in justifying our beliefs about mathematical truths. In essence, four conditions must be met in acquiring mathematical knowledge: justification, truth, belief, and *provability*. It showed dilemma to abstract knowledge and proposed that a similar *provability*-like property should be introduced to form a causal connection between truth, justification, and belief. hopefully this study will stimulate new ideas and broaden perspectives on these issues.

References

[1] Gulley, Norman. Plato's theory of knowledge. Routledge, 2013.

[2] Parikh, Rohit, and Adriana Renero. "Justified true belief: plato, gettier, and turing." Philosophical Explorations of the Legacy of Alan Turing: Turing 100 (2017): 93-102.

[3] Whitesmith, Martha. "Justified true belief theory for intelligence analysis." Intelligence and National Security 37.6 (2022): 835-849.

[4] Turri, John. "In Gettier's wake." Epistemology: The key thinkers (2012): 214-229.

[5] Gettier, Edmund. "Is justified true belief knowledge?." Arguing about knowledge. Routledge, 2020. 14-15.

[6] Goldman, Alvin. "A causal theory of knowing." en (5), pgs (1976): 138-153.

[7] Benacerraf, Paul. "Mathematical truth." The Journal of Philosophy 70.19 (1973): 661-679.

[8] Noonan, Harold. Routledge Philosophy Guidebook to Kripke and Naming and Necessity. Routledge, 2014.

[9] Beziau, Jean-Yves. "What is an axiom?." A True Polymath-A Tribute to Francisco Antonio Doria, College Publications, London (2020): 122-142.

[10] Price, Huw. "Why 'not'?." Mind 99.394 (1990): 221-238.